from which the representation of $p(x)$ follows.

With u and v defined as in the foregoing solution, we have that x is a root of $p(x)$ if and only if $p_{n+1}(x) - p_n(x) = 0$, equivalent to

$$\begin{aligned} 0 &= u^{n+1} + v^{n+1} - u^n - v^n = v^{n+1}(u^{2n+2} + 1 - u^{2n+1} - u) \\ &= v^{n+1}(u^{2n+1} - 1)(u - 1). \end{aligned}$$

Since $u \neq 0, 1$, then u is a nontrivial $(2n+1)^{\text{th}}$ root of unity and we can finish off as in Solution 2.

Editor's comments. The proposer found that $(t - 1/t)q(t) = (t^{2n+1} - t^{-(2n+1)})$, noted that $q(t) = 0$ whenever $t^{2n+1} = 1$, and obtained the set of roots of $p(x)$ in the form

$$\left\{2i \sin\left(\frac{2k\pi}{2n+1}\right) : k = 1, 2, \ldots, 2n\right\}.$$

4249. *Proposed by Daniel Sitaru.*

Let a,b,c be real numbers with at most one of them equal to zero. Prove that

$$\frac{(a-b)^2(b-c)^2(c-a)^2}{a^2b^2 + b^2c^2 + c^2a^2} \leq 2(a^2 + b^2 + c^2 - ab - bc - ca).$$

There were twelve correct solutions. Five of them applied a standard inequality; the first solution gives a sample. Four solvers used symmetric functions; while most used a discriminant condition on either a quadratic or cubic to obtain the inequality, the solver of our third solution kept the complications to a minimum. An additional solver used Maple.

Solution 1, by Mihai Bunget; Dionne Bailey, Elsie Campbell and Charles Diminnie; Oliver Geupel; and Kevin Soto Palacios (independently).

The inequality is trivial if one of a, b, c vanishes. Otherwise, observe that

$$(a-b)(b-c)(c-a) = ab(b-a) + bc(c-b) + ca(a-c)$$

and that

$$2(a^2 + b^2 + c^2 - ab - bc - ca) = (b-a)^2 + (c-b)^2 + (a-c)^2.$$

The result follows directly from the Cauchy-Schwarz Inequality applied to the vectors (ab, bc, ca) and $(b-a, c-b, a-c)$. Equality occurs if and only if $a = b = c$.

Solution 2, by Titu Zvonaru.

With the convention that each sum is cyclic in a, b, c over three terms, we have

$$\begin{aligned}
&2(a^2+b^2+c^2-ab-bc-ca)(a^2b^2+b^2c^2+c^2a^2)-(a-b)^2(b-c)^2(c-a)^2\\
&\quad=[2\sum a^4b^2+2\sum a^2b^4+6a^2b^2c^2-2\sum a^3b^3-2\sum ab^2c^3-2\sum a^3b^2c]\\
&\qquad-[\sum a^4b^2+\sum a^2b^4+2\sum ab^2c^3+2\sum a^3b^2c\\
&\qquad-2\sum a^3b^3-2\sum a^4bc-6a^2b^2c^2]\\
&\quad=\sum a^4b^2+\sum a^2b^4+12a^2b^2c^2+2\sum a^4bc-4\sum a^3b^2c-4\sum ab^2c^3\\
&\quad=\sum a^2(ab+ac-2bc)^2\ge 0,
\end{aligned}$$

from which the inequality follows, with equality when $a = b = c \neq 0$.

Solution 3, by Arkady Alt.

The inequality holds when one of a, b, c vanishes. Assume $abc \neq 0$. Since the inequality is homogeneous, we may assume that $a+b+c = 1$. Let $u = ab+bc+ca$ and $v = abc$. Then

$$a^2+b^2+c^2-ab-bc-ca=1-3u,$$

$$a^2b^2+b^2c^2+c^2a^2=u^2-2v,$$

and

$$(a-b)^2(b-c)^2(c-a)^2=u^2-4u^3+18uv-4v-27v^2.$$

Hence

$$\begin{aligned}
&2(a^2+b^2+c^2-ab-bc-ca)(a^2b^2+b^2c^2+c^2a^2)-(a-b)^2(b-c)^2(c-a)^2\\
&\quad=2(1-3u)(u^2-2v)-(u^2-4u^3+18uv-4v-27v^2)\\
&\quad=u^2-2u^3-6uv+27v^2\\
&\quad=\frac{(9v-u)^2}{3}+\frac{2u^2(1-3u)}{3}.
\end{aligned}$$

Since

$$1-3u=(a+b+c)^2-3(ab+bc+ca)=(1/2)[(a-b)^2+(b-c)^2+(c-a)^2]\ge 0,$$

the right side of the equation is nonnegative, as desired.

Editor's comments. The proposer took an entirely different approach. The matrix $A = \begin{pmatrix} B \\ C \end{pmatrix}$ with

$$B=\begin{pmatrix}1 & 1 & 1\\ a & b & c\end{pmatrix}\quad\text{and}\quad C=\begin{pmatrix}\frac{1}{a} & \frac{1}{b} & \frac{1}{c}\end{pmatrix}$$

is a 3×3 matrix. We note that

$$\det(B\cdot B^T)=2(a^2+b^2+c^2-ab-bc-ca),$$